Eur. Phys. J. C 5, 501–506 (1998)
DOI 10.1007/s100529800869

THE EUROPEAN
PHYSICAL JOURNAL C


# Axial anomaly and low energy tests for instanton vacuum models in QCD

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Received: 24 November 1997 / Published online: 10 March 1998

**Abstract.** We propose to test instanton models of QCD – in particular the single instanton approximation and the Diakonov and Petrov model – against some exact relations inferred from QCD + QED axial anomaly. One of these relations, in chiral limit, is satisfied by the latter model, but not by the former one. More refined tests, obtained beyond the chiral limit, are not fulfilled by the Diakonov and Petrov model.

## 1 Introduction

As is well-known, quantum fluctuations may destroy the symmetry of the classical lagrangian [1–3]. In gauge theories the most important examples of this kind are the axial anomalies in electroweak theory [2,3] and in QCD [4,5]. The axial anomaly arises from noninvariance of the fermionic measure against axial transformations in the path integrals of the theory [6] (see also [7], concerning higher-loop corrections). In Euclidean QCD+QED the axial anomaly reads

$$\partial_\mu j^5_\mu = -iN_f \frac{g^2}{16\pi^2} G\tilde{G} - iN_c \frac{e^2}{8\pi^2} \sum_f Q_f^2 F\tilde{F} + 2\sum_f m_f \psi_f^\dagger \gamma_5 \psi_f, \tag{1}$$

where $j^5_\mu = \sum_f \psi_f^\dagger \gamma_\mu \gamma_5 \psi_f$ $(f = u, d, s)$ is the quark singlet axial current, $\psi_f$ the quark field, $N_f$ the number of the light flavors, $g$ the QCD coupling constant, $2G\tilde{G} = \epsilon^{\mu\nu\lambda\sigma} G^a_{\mu\nu} G^a_{\lambda\sigma}$, $G^a_{\mu\nu}$ the gluon field strength operator, $N_c$ the number of colours, $e$ the QED coupling constant, $Q_f$ the electric fractional charges of the quarks, $F_{\mu\nu}$ the photon field strength operator. We have explicitly included the contributions of the current masses of light quarks $m_f$.

Although the anomaly equations have been deduced in the framework of perturbation theory [2,3] (see also [8, 9]), (1) has very important applications at low energies [4,5,10–12], where nonperturbative solutions – especially instantons [4] – play an essential role. In particular it has been pointed out (see review [5] and references therein) that this equation leads to the nontrivial low-energy theorem ($LET$) (2) and to the useful relations (3), (4) (see below), which may be used as a test for instanton vacuum models [13].

In the present article we consider further tests for such models by means of the matrix elements of (1) ($i$) between vacuum and two-photons states and ($ii$) between vacuum and meson states.

As to case ($i$), nonvanishing of the $\eta'$ meson mass $m_{\eta'}$ even in chiral limit (due to axial anomaly) and an accurate tensor analysis [14] imply that for real photons the matrix element of the divergence of the singlet axial current vanishes in the $q^2 << m^2_{\eta'}$ limit, giving rise to the following $LET$:

$$\langle 0|N_f \frac{g^2}{16\pi^2} G\tilde{G}|2\gamma\rangle = N_c \frac{e^2}{4\pi^2} \sum_f Q_f^2 F^{(2)} \tilde{F}^{(3)} - 2i \sum_f m_f \langle 0|\psi_f^\dagger \gamma_5 \psi_f|2\gamma\rangle, \tag{2}$$

where $F^{(i)}_{\mu\nu} = \epsilon_{i,\mu} q_{i,\nu} - \epsilon_{i,\nu} q_{i,\mu}$ and $q_i$, $\epsilon_i$ $(i = 2, 3)$ are respectively the momenta and polarization vectors of photons and $q = q_2 + q_3$. Equation (2) is an exact low energy relation, which cannot be fulfilled in the framework of perturbation theory. In this case the l.h.s. of (2) is of order $g^4 e^2$, since gluons can couple to photons only by virtual quarks. On the contrary at the r.h.s. the first term does not contain any strong coupling at all, nor any compensation occurs with the second term, because this latter vanishes in the chiral limit [5]. Only a nonperturbative contribution of order $g^{-2}$ – as the one provided by instantons [15] – may cancel the factor $g^2$ at the l.h.s..

Further stringent tests for instanton vacuum models can be obtained from the matrix elements of (1) between

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vacuum and meson states (case $(ii)$), i.e.,

$$\langle 0|N_f \frac{g^2}{16\pi^2} G\tilde{G}|\eta\rangle = -2im_s \langle 0|\psi_s^\dagger \gamma_5 \psi_s|\eta\rangle, \qquad (3)$$

$$\langle 0|N_f \frac{g^2}{16\pi^2} G\tilde{G}|\pi^0\rangle = -i(m_u - m_d)\langle 0|\psi^\dagger \tau_3 \gamma_5 \psi|\pi^0\rangle. \qquad (4)$$

In order to calculate the l.h.s. of (2), we must consider the three-point function

$$\tau_{\mu\nu}(x_1, x_2, x_3) = \langle 0|T[g^2 G\tilde{G}(x_1) j_\mu^{em}(x_2) j_\nu^{em}(x_3)]|0\rangle, \quad (5)$$

($j_\mu^{em}$ being the electromagnetic current). In particular, according to our preceding observation, we need the instanton contribution to $\tau_{\mu\nu}$ in the Euclidean space of the three-point function, which contains all information on the matrix element $\langle 0|G\tilde{G}|2\gamma\rangle$ and also part of information concerning $\eta \to 2\gamma$ decay, $2\gamma$- transitions in heavy quarkonia etc.

Instantons – whose presence in QCD is a well established fact, at least at a phenomenological level and in numerical simulations of QCD vacuum [18] – constitute the main input to our calculations, especially in connection with quark propagation. To our present knowledge the instanton structure of QCD vacuum is concentrated in an average size $\rho$ and in an average interinstanton distance $R$, such that [16,17]

$$\rho = 1/3fm, \quad R = 1fm. \qquad (6)$$

Therefore the packing parameter $(\rho/R)^4 = 0.012$ is small, legitimating independent averaging over positions and orientations of the instantons; moreover in some cases it is even possible to use the quark propagator in the single-instanton field [20].

In a previous work [13] $LET$ was used as a test for the Diakonov-Petrov ($DP$) ansatz [21,23] of the low-energy QCD effective action in the chiral limit. The ansatz, based on an interpolation formula for the quark propagator in the field of a single instanton, does satisfy $LET$ to $\sim 17\%$ accuracy.

The aim of the present article is to test, first of all, one-instanton approximation against $LET$ in the chiral limit and, secondly, the $DP$ model against more refined tests beyond the chiral limit.

In Sect. 2 we calculate $\tau_{\mu\nu}$ in single instanton approximation. In Sect. 3 we rederive the $DP$ effective action starting from results of Lee&Bardeen [24] ($LB$) and accounting for current quark masses $m_f$. In Sect. 4 we calculate $\tau_{\mu\nu}$ by this effective action and check it against $LET$, both in chiral limit and beyond this approximation. In Sect. 5 we calculate in the same framework both sides of relations (3) and (4) and compare each l.h.s. with its respective r.h.s.

## 2 Single-instanton approximation

A lot of papers [25–27] have been devoted to instanton calculus in single-instanton approximation, where the calculation of $\tau_{\mu\nu}$ amounts to computing integrals like

$$\int d^4 z_\pm dU_\pm \langle T(g^2 G\tilde{G}(x_1) j_\mu^{em}(x_2) j_\nu^{em}(x_3)\rangle_\pm$$
$$= \sum_f Q_f^2 \int d^4 z_\pm (G\tilde{G}(x_1)_\pm$$
$$\times Tr(\gamma_\mu S_\pm(x_2, x_3)\gamma_\nu S_\pm(x_3, x_2)). \qquad (7)$$

Here $z_\pm$ and $U_\pm$ are the position and orientation of the (anti-)instanton, assuming instanton integration sizes to be peaked at $\rho$, (6); moreover

$$(g^2 G\tilde{G}(x))_\pm = \pm f(x-z) = \pm \frac{192\rho^4}{[\rho^2 + (x-z)^2]^4} \qquad (8)$$

and $S_\pm = (i\hat{D}_\pm + im)^{-1}$ is the full quark propagator in presence of a single (anti)instanton.

Starting from the expression of $S_\pm$ given in [20], we obtain

$$\tau_{\mu\nu}(x_1, x_2, x_3)$$
$$= \frac{N}{V}\frac{192 N_c \rho^6}{3\pi^4} \sum_f Q_f^2 \int d^4 z h^4(x_1 - z)$$
$$\times \frac{h(x_2 - z)h(x_3 - z)}{(x_2 - x_3)^2}$$
$$\times \left[\frac{h(x_2 - z) + h(x_3 - z)}{(x_2 - x_3)^2} + h(x_2 - z)h(x_3 - z)\right]$$
$$\times \epsilon_{\mu\nu\alpha\beta}(x_2 - z)_\alpha (x_3 - z)_\beta, \qquad (9)$$

where $h(x) = (\rho^2 + x^2)^{-1}$. The correlator, that is the Fourier transform $\hat{\tau}_{\mu\nu}$ of the three-point function $\tau_{\mu\nu}$, which has to be tested against $LET$, results to be

$$\hat{\tau}_{\mu\nu}(q_1, q_2, q_3)$$
$$= \frac{N}{V}\frac{N_c \rho^2}{3\pi^4} \sum_f Q_f^2 (2\pi)^4 \delta(\sum_i q_i) \hat{f}(q_1^2)$$
$$\times 4! \int_0^1 a da \int_0^1 db (a(1 - a + ab(1-b))^{-3/2}$$
$$\times \epsilon_{\mu\nu\alpha\beta}(-\frac{\partial^2}{\partial p_{2,\alpha}\partial p_{3,\beta}}) J(P), \qquad (10)$$

as can be shown by applying the Feynman integration technique. We have set

$$\hat{f}(q_1^2) = \int d^4 x_1 \, \frac{192\rho^4}{(\rho^2 + x_1^2)^4} \, \exp(iq_1 x_1),$$
$$J(P) = \int d^8 Y \, [Y^2 + r^2]^{-5} \, \exp iPY, \qquad (11)$$

having introduced the 8-dim vectors $P = (p_2, p_3)$, $Y = (y_2, y_3)$, with

$$p_2 = \frac{q_2 + bq_3}{(1 - a + ab(1-b))^{1/2}}, \; p_3 = \frac{q_3}{a^{1/2}}, \; r^2 = \rho^2(1 - ab).$$

The latter integral (11) can be easily calculated and is reduced to the MacDonald function, i.e.,

$$J(P) = \frac{\pi^4}{4!}\frac{P}{r}K_1(Pr),\; P = (P^2)^{1/2}. \quad (12)$$

In the limit of small momenta

$$K_1(Pr) = \frac{1}{Pr} + \frac{Pr}{2}[\ln\frac{Pr}{2} + C - 1/2] + ..., \quad (13)$$

where $C = 0.577215...$ is the Euler constant.

Equations (10), (12), (13) imply that $\hat{\tau}_{\mu\nu}(q_1, q_2, q_3)$ is divergent for $q_i^2 \to 0$, that is, the model badly violates $LET$. So does the improved single-instanton approximation [18], where the zero-mode contribution to the propagator has been modified by replacing $m$ by $m^*$, the effective mass of the quark, accounting for the influence of the surrounding instantons. We conclude that single-instanton approximation is not suitable for calculating three-point functions at small momenta, owing to the slow decrease of $\tau_{\mu\nu}$ at large distances. Indeed we have neglected rescattering effects of quarks by other instantons, which, during quark propagation, lead to the formation of the constituent quark, producing a suitable effective mass and providing needed exponential decrease with distance. Such effects – which, by the way, do not affect appreciably the two-point functions of vector currents – can be described by an effective action, as we are going to show.

## 3 A derivation of the $DP$ effective action

It is natural to choose the singular gauge for the instantons in describing many instanton effects in the propagation of the quarks. In the case of a small packing parameter it is possible to do the following ansatz for the background instanton field:

$$A_\mu(x) = \sum_{+}^{N_+} A_{+,\mu}(x;\xi_+) + \sum_{-}^{N_-} A_{-,\mu}(x;\xi_-),$$
$$(\xi_\pm = (z_\pm, U_\pm, \rho_\pm)), \quad (14)$$

where, $z_i$, $U_i$ and $\rho_i$ the position, orientation and size of the $i$-th instanton. The canonical partition function of the $N_+$instantons and $N_-$ antiinstantons can be schematically written as

$$Z_{N_+,N_-} = \int det_N \exp(-V_g) \prod_i^{N_+,N_-} d^4z_i dU_i dn(\rho_i), \quad (15)$$

where $V_g$ is the instanton-(anti)instanton interaction potential generated by the gluon field action and $det_N$ is a quark determinant in the instanton field. The main assumption of the instanton model is that $V_g$ is repulsive at small distances between instanton and antiinstanton. This should provide the stabilization of the instanton sizes and of the interinstanton distances. We mainly deal with $det_N$, which describes the influence of light quarks.

Lee and Bardeen [24] ($LB$) calculated the quark propagator in a more sophisticated approximation than single-instanton, finding

$$det_N = detB,\; B_{ij} = im\delta_{ij} + a_{ji}, \quad (16)$$

where $a_{ij}$ is the overlapping matrix element of the quark zero-modes $\Phi_{\pm,0}$ generated by instantons. This matrix element is nonzero only between instantons and antiinstantons (and vice versa) due to the chiral factor in $\Phi_{\pm,0}$, i.e.,

$$a_{-+} = - < \Phi_{-,0}|i\hat{\partial}|\Phi_{+,0} > . \quad (17)$$

The overlapping of the quark zero-modes causes quark jumping from one instanton to another one during propagation.

Equation (16) implies that for $N_+ \neq N_-$ $det_N \sim m^{|N_+ - N_-|}$, so the fluctuations of $|N_+ - N_-|$ are strongly suppressed due to presence of light quarks. Therefore we assume $N_+ = N_- = N/2$.

Following the procedure suggested in [28], we get the fermionization of $LB$'s result, i.e.,

$$det_N = \int D\psi D\psi^\dagger \exp(\int d^4x \sum_f \psi_f^\dagger i\hat{\partial}\psi_f)$$
$$\times \prod_f (\prod_+^{N_+}(im_f + V_+[\psi_f^\dagger, \psi_f])$$
$$\times \prod_-^{N_-}(im_f + V_-[\psi_f^\dagger, \psi_f])), \quad (18)$$

where

$$V_\pm[\psi_f^\dagger, \psi_f] = \int d^4x(\psi_f^\dagger(x) i\hat{\partial}\Phi_{\pm,0}(x;\xi_\pm))$$
$$\times \int d^4y(\Phi_{\pm,0}^\dagger(y;\xi_\pm) i\hat{\partial}\psi_f(y)). \quad (19)$$

Equation (18) coincides with the ansatz for the fixed $N$ partition function postulated by $DP$, except for the sign in front of $V_\pm$. Keeping in mind the low density of the instanton media, which allows independent averaging over positions and orientations of the instantons, (18) leads to the partition function

$$Z_N = \int D\psi D\psi^\dagger \exp(\int d^4x\, \psi^\dagger i\hat{\partial}\psi)\, W_+^{N_+} W_-^{N_-}, \quad (20)$$

where

$$W_\pm = \int d^4\xi_\pm \prod_f (V_\pm[\psi_f^\dagger \psi_f] + im_f)$$
$$= (-i)^{N_f} \left(\frac{4\pi^2\tilde{\rho}^2}{N_c}\right)^{N_f}$$
$$\times \int \frac{d^4z}{V} \det_f (iJ_\pm(z) - \frac{mN_c}{4\pi^2\rho^2}) \quad (21)$$

and

$$J_\pm(z)_{fg} = \int \frac{d^4k d^4l}{(2\pi)^8} \exp(-i(k-l)z)\, F(k^2) \times F(l^2)\, \psi_f^\dagger(k) \tfrac{1}{2}(1\pm\gamma_5)\psi_g(l). \quad (22)$$

The form factor $F$ is related to the zero–mode wave function in momentum space $\Phi_\pm(k;\xi_\pm)$ and is equal to

$$F(k^2) = -t\frac{d}{dt}\left[I_0(t)K_0(t) - I_1(t)K_1(t)\right], \quad t = \frac{1}{2}\sqrt{k^2}\rho. \quad (23)$$

## 4 Correlator in the $DP$ effective action

In quasiclassical (saddle point) approximation any gluon operator derives its main contribution from instanton background. In the following the operator $g^2G\tilde{G}(x)$ will be considered. Owing to the low density of the instanton medium, it is possible to neglect the overlap of the fields of different instantons. In that case, the matrix element of $g^2G\tilde{G}(x)$ with any other quark operator $Q$ is

$$\begin{aligned}\langle g^2G\tilde{G}(x)Q\rangle_N &= Z_N^{-1}\int D\psi D\psi^\dagger \exp(\int d^4x\psi^\dagger i\hat{\partial}\psi) \\ &\times \Big(N_+\left(W_{G\tilde{G}+}(x)Q\right)W_+^{N_+-1}W_-^{N_-} \\ &+N_-\left(W_{G\tilde{G}-}(x)Q\right)W_+^{N_+}W_-^{N_--1}\Big), \quad (24)\end{aligned}$$

where

$$W_{G\tilde{G}\pm} = \pm\left(\frac{4\pi^2\tilde{\rho}^2}{N_c}\right)^{N_f}\int\frac{d^4z}{V}\, f(x-z) \times \det_f(J_\pm(z) + i\frac{mN_c}{4\pi^2\rho^2}). \quad (25)$$

It is useful to introduce the external fields $\kappa(x)$ and $a(x)$, coupled respectively to $g^2G\tilde{G}$ and to the electromagnetic quark current. Starting from (25), we find the partition function $\hat{Z}[\kappa,a]$ describing mesons [13] in presence of such external fields:

$$\hat{Z}[\kappa,a] = \int D\Phi_+ D\Phi_- exp(-W[\Phi_+,\Phi_-]), \quad (26)$$

where

$$\begin{aligned}W[\Phi_+,\Phi_-] &= \int d^4x(w_a+w_b-w_c),\\ w_a &= (N_f-1)\frac{N}{2V}(\prod_f M_f^{-1}det\Phi_+)^{(N_f-1)^{-1}} + (+\to -),\\ w_b &= \frac{N_c}{4\pi^2\rho^2}Tr(m(\Phi_++\Phi_-)),\\ w_c &= \sum_f Trln\frac{i\hat{D}+iF^2(\Phi_+\beta_++\Phi_-\beta_-)}{i\hat{\partial}+im_f},\\ \hat{D} &= \hat{\partial}-ieQ\hat{a},\ \beta_\pm = \left[(1\pm(\kappa f))^{N_f^{-1}}\right]\frac{1}{2}(1\pm\gamma_5).\end{aligned} \quad (27)$$

The saddle point of the integral (26) is located at $(\Phi_\pm)_{fg} = M_f\delta_{fg}$, a self-consistency condition for the effective quark mass, i.e.,

$$4N_cV\int\frac{d^4k}{(2\pi)^4}\frac{M_f^2F^4(k^2)}{M_f^2F^4(k^2)+k^2} = N + \frac{m_fM_fVN_c}{2\pi^2\rho^2}, \quad (28)$$

being imposed, which describes also the shift of the effective mass of the quark $M_f$ due to current mass $m_f$. Expanding (28) with respect to $m_f$ yields $M_f = M_0 + \gamma m_f$, where

$$\gamma^{-1} = \rho^2\int_0^\infty dk^2\frac{k^4F^4(k^2)}{(M_0^2F^4(k^2)+k^2)^2}. \quad (29)$$

Such integrals converge due to the form factor $F(k^2)$. Assuming for the parameters $\rho$ and $R$ the values (6) – which correspond to $M_0 = 340\,MeV$ – we find

$$\gamma = 2.4. \quad (30)$$

The quark condensate is then given by

$$-V^{-1}Z_N^{-1}\frac{dZ_N}{dm}|_{\kappa=0} = -\frac{N_cM_0}{2\pi^2\rho^2} = -(265\,MeV)^3 \sim N_c\frac{1}{\rho R^2}. \quad (31)$$

This quantity can be also calculated by formula [21] $-i <\psi^\dagger\psi>_{Euclid} = iTr\,S$, yielding

$$-i<\psi^\dagger\psi>_{Euclid} = -4N_c\int\frac{d^4k}{(2\pi)^4}\frac{M_0F^2(k^2)}{M_0^2F^4(k^2)+k^2} = -(255\,MeV)^3 \sim N_c\frac{1}{\rho R^2}. \quad (32)$$

Although coming from different formulas, predictions (31) and (32) have the same parametric dependence and agree with the phenomenological value, i.e.,

$$-i<\psi^\dagger\psi>_{Euclid} = -(240-250\,MeV)^3.$$

The three-point function can be derived from the functional relation

$$\tau_{\mu\nu}(x_1,x_2,x_3) = \frac{\delta\hat{W}[\kappa,a]}{\delta\kappa(x_1)\delta a_\mu(x_2)\delta a_\nu(x_3)}|_{\kappa,a=0}, \quad (33)$$

where [13]

$$\hat{W}[\kappa,a] = \sum_f Trln\frac{i\hat{D}+iM_fF^2(\beta_++\beta_-)}{i\hat{\partial}+im_f}. \quad (34)$$

Equation (34) implies that the vertex factors in the diagram of the process are $eQ_f\gamma_\mu$ and $iM_ffF^2N_f^{-1}\gamma_5$. Taking the Fourier transform of (33), we get

$$\hat{\tau}_{\mu\nu}(q_1,q_2,q_3) = \hat{f}(q_1^2)N_ce^2\sum_f Q_f^2 8M_f^2\epsilon^{\mu\nu\lambda\sigma}q_{2\lambda} \times q_{3\sigma}\Gamma_f(q_1^2,q_2^2,q_3^2), \quad (35)$$

where $\Gamma_f(q_1^2, q_2^2, q_3^2)$, the form factor coming from the diagram of the process considered, may be calculated analytically if we approximate the form factor $F$ by 1. As a result, the l.h.s. of (2) reduces to

$$(N_f \frac{g^2}{16\pi^2})(\frac{4e^2 N_c}{g^2 N_f} \sum_f Q_f^2) F^{(2)} \tilde{F}^{(3)}, \tag{36}$$

which coincides with the first term at the r.h.s. of (2). If we take into account the form factor $F$ in (35), and give the model parameters the values (6), in the chiral limit we find [13] a variation of $\sim 17\%$.

Beyond the chiral limit the l.h.s. of (2) receives the contribution

$$-0.2\gamma\rho \frac{N_c e^2}{2\pi^2} \sum_f Q_f^2 m_f \epsilon^{\mu\nu\lambda\sigma} q_{2\lambda} q_{3\sigma}. \tag{37}$$

This quantity has to be compared with the explicit contribution of the current quark masses to the r.h.s. of (2), which, too, may be calculated by formulas (34) and (33) substituting $iM_f f F^2 N_f^{-1}$ by $2im_f$ in the $\gamma_5$-vertex. Approximating again $F \sim 1$, we obtain

$$\begin{aligned}-2i \sum_f m_f \langle 0|\psi_f^\dagger \gamma_5 \psi_f|2\gamma\rangle \\ = -\frac{N_c e^2}{2\pi^2} \sum_f Q_f^2 \frac{m_f}{M_f} \epsilon^{\mu\nu\lambda\sigma} q_{2\lambda} q_{3\sigma}.\end{aligned} \tag{38}$$

The ratio of (37) to (38) at the parameter values (6) results to be

$$0.2\,\gamma\,\rho\,M_0 \;=\; 0.28 \tag{39}$$

and not 1, as demanded by $LET$. We stress the net contradiction with the theorem, not only in magnitude but also in the sign. So the $DP$ model (20) fails to reproduce $LET$ beyond chiral limit.

## 5 Further tests for instanton models

The matrix elements of the anomaly (1) between vacuum and $\eta$-meson or $\pi^0$ states lead to more stringent tests of the $DP$ model.

The partition function (27) describes mesons with matrices $\Phi_\pm$, whose usual decomposition is [29]

$$\Phi_\pm = \exp(\pm\frac{i}{2}\phi) M\sigma \exp(\pm\frac{i}{2}\phi), \tag{40}$$

$\phi$ and $\sigma$ being $N_f \times N_f$ matrices. The saddle-point condition demands $\sigma = 1$, $\phi = 0$. The usual decomposition for the pseudoscalar fields $\phi = \sum_0^8 \lambda_i \phi_i$ may be used, where $Tr\lambda_i\lambda_j = 2\delta_{ij}$ and $\phi_{8(3)}$ can be identified with the $\eta(\pi^0)$-meson state.

Firstly we consider the matrix element in which the $\eta$-meson is involved. Neglecting the small admixture factor ($\sim 0.1$) with the pure singlet state [30], the matrix element of the divergence (2) between the $\eta$-meson and the vacuum leads to (3), which can be used as a test for instanton models.

From previous considerations it follows that the factor $g^2 G\tilde{G}$ generates the vertex $iMfF^2\gamma_5 N_f^{-1}$ and the $\eta$-meson gives rise to $iM\lambda_8 F^2\gamma_5$. The structure of the mass matrix is

$$M \;=\; M_0 \;+\; \gamma(m_s(\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8) + m_u\frac{1+\tau_3}{2} + m_d\frac{1-\tau_3}{2}).$$

Then at small $q$

$$\begin{aligned}\langle 0|N_f \frac{g^2}{16\pi^2} G\tilde{G}|\eta\rangle = 2\gamma m_s(-\frac{1}{\sqrt{3}} tr(\lambda_8)^2) 4N_c \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{M_0 F^4(k^2)}{M_0^2 F^4(k^2) + k^2}.\end{aligned} \tag{41}$$

Applying the same procedure that led to (28), we get

$$\begin{aligned}&\langle 0|N_f \frac{g^2}{16\pi^2} G\tilde{G}|\eta\rangle \\ &= 2\gamma m_s(-\frac{2}{\sqrt{3}})\frac{N}{VM_0} \sim m_s \frac{N_c^{1/2}}{\rho R^2}.\end{aligned} \tag{42}$$

The r.h.s. of (3) is

$$\begin{aligned}&-2im_s\langle 0|\psi_s^\dagger\gamma_5\psi_s|\eta\rangle \\ &= -2m_s(-\frac{2}{\sqrt{3}})4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M_0 F^2(k^2)}{M_0^2 F^4(k^2) + k^2} \\ &\sim m_s \frac{N_c^{1/2}}{\rho R^2}.\end{aligned} \tag{43}$$

On the other hand (32) yields

$$-2im_s\langle 0|\psi_s^\dagger\gamma_5\psi_s|\eta\rangle = -2m_s(-\frac{2}{\sqrt{3}}) i < \psi^\dagger\psi > . \tag{44}$$

Now let us confront such equations with some consequences of the relation (4), concerning the $\pi^0$-meson. Repeating for this case the calculations applied to relation (3), the l.h.s. of (4) yields

$$\langle 0|N_f \frac{g^2}{16\pi^2} G\tilde{G}|\pi^0\rangle = 2\gamma(m_u - m_d)\frac{N}{VM_0}, \tag{45}$$

while the r.h.s. results in

$$\begin{aligned}&-2i(m_u - m_d)\langle 0|\psi^\dagger \frac{\tau_3}{2}\gamma_5\psi|\pi^0\rangle \\ &= -2(m_u - m_d) i < \psi^\dagger\psi > .\end{aligned} \tag{46}$$

The ratio of (42) to (44) equals the ratio of (45) to (46) and at the parameter values (6) it yields

$$\frac{N}{VM_0 i < \psi^\dagger\psi >} = -0.66 \tag{47}$$

and not 1, as demanded by relations (3) and (4). We stress the net contradiction with the theorem, not only in the magnitude, but also in the sign. Again the $DP$ model (20) strongly contradicts some consequences of the operator (1) beyond the chiral limit.

## 6 Conclusions

We recall the main results of our treatment. The *LET* in chiral limit discriminates between the single-instanton approximation and the more sophisticated *DP* ansatz. More refined tests beyond chiral limit are violated even by this model. All the above illustrated tests could be used as guides for building more and more satisfactory instanton trial solutions.

*Acknowledgements.* The work of M. Musakhanov is supported in part by the NATO-CNR grant and by grant INTAS-93-239ext. One of the authors (M.M.) is very grateful to the Department of Physics of the University of Genova for its hospitality.

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